J Glob Optim (2010) 46:571–580
DOI 10.1007/s10898-009-9441-z

A field guide to the matrix classes found in the literature of the linear complementarity problem

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Received: 7 May 2009 / Accepted: 7 May 2009 / Published online: 31 May 2009

Abstract There are more than 50 matrix classes discussed in the literature of the Linear Complementarity Problem. This guide is offered as a compendium of notations, definitions, names, source information, and commentary on these many matrix classes. Also included are discussions of certain properties possessed by some (but not all) of the matrix classes considered in this guide. These properties—fullness, completeness, reflectiveness, and sign-change invariance—are the subject of another table featuring matrix classes that have one or more of them. Still another feature of this work is a matrix class inclusion map depicting relationships among the matrix classes listed herein.

Keywords Linear complementarity problem · Matrix classes · Structural properties · Inclusion map

1 Introduction

This paper is meant to serve as a useful guide to the scores of matrix classes that populate the literature of the linear complementarity problem (LCP). Unconventional though the paper may be, parts of it do have precedents, several of which are cited below. At this point, it is my pleasure to acknowledge that two excellent conferences organized by Professor Franco Giannessi and held at the "E. Majorana" Centre for Scientific Culture (in Erice, Sicily) stimulated my own efforts to assemble information of the sort contained in the present article.

Awareness of the importance of matrix classes and their properties in the LCP developed with that of the problem itself. It must be said, however, that the present-day abundance of distinct matrix classes was not foreseen in the early days of the subject. Eaves [15] was

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probably the first to assign notations to the matrix classes he considered in his Ph.D. thesis on the linear complementarity problem. This practice was carried forward by Lemke [27] and Parsons [31]. Soon thereafter (1972), Karamardian [23] assigned names to some classes that hitherto had been known only by their notations. He also depicted the inclusions that were known to hold among the 12 matrix classes that had been identified in the literature. Today, with more than 50 matrix classes to contend with, it is not easy for one to keep them all in mind. Accordingly, this guide is offered as a compendium of notations, definitions, names, source information, and commentary on these many matrix classes. An additional noteworthy feature of this work is a matrix class inclusion map depicting relationships among the matrix classes listed herein.

In this work, after a few preliminaries, there appears a list of matrix classes, their definitions, their common names (if any), their source in the literature, and, in some cases, a note. A bibliography at the end of the paper is included to facilitate the tracing of the sources of the ideas presented. In a few cases, for instance, the notes attempt to untangle the complicated history of certain matrix classes, especially with regard to their notation. In at least a few cases, it has been discovered that new class-definitions and notations do not actually lead to new classes. These, too, are pointed out in the notes.

Also included in the notes are discussions of certain properties possessed by some (but by no means all) of the matrix classes considered in this guide. These properties—fullness, completeness, reflectiveness, and sign-change invariance—are the subject of another table featuring matrix classes that have one or more of them.

Over the years, many authors have compiled lists of matrix classes related to the LCP, and a few authors have created diagrams indicating the inclusion relationships between most of the matrix classes known at the time. Figure 5.1 in Richard Stone's Ph.D. thesis [33] was the most comprehensive as of 1981. The figure provided in the present document modifies and extends Stone's by including the many classes that have been identified since then.

I have tried to make this catalog of matrix classes as exhaustive as possible, but I also had to impose some limits. For example, the intersection of two (or more) matrix classes occasionally gives a new class with interesting properties. Some are included in the list; but it would be impractical (if not foolish) to consider the intersection of *every* pair of these classes. A similar statement can be made for instances where one class is included in another and their set-theoretic difference then becomes an object of study. With one exception, I have also chosen to not to include the classes of the "almost-this-or-that-matrix-class" type.

2 A few preliminaries

A given vector $q \in \mathcal{R}^n$ and matrix $M \in \mathcal{R}^{n\times n}$ determine a system (q, M) of linear inequalities and a nonlinear equation called a linear complementarity problem (LCP). That system is

$$x \geq 0, \tag{1}$$

$$q + Mx \geq 0, \tag{2}$$

$$x^{\mathrm{T}}(q + Mx) = 0. \tag{3}$$

The LCP asks for a solution or evidence that no solution exists. An algorithm that accomplish this is said to *process* the problem.

When conditions (1) and (2) have a solution, they (and the problem) are said to be *feasible*. The problem is *solvable* if there exists a vector x simultaneously satisfying all three conditions. The notion of solvability is, in principle, independent of algorithms that solve the problem, but in some cases, solvability is actually established by means of one or more algorithms. For some choices of q and M, the LCP (q, M) can be feasible but not solvable. Thus, an algorithm that processes (q, M) must be able to detect the nonexistence of a solution. Fortunately, there are some matrix classes having the property that feasibility implies solvability. Accordingly, the following two sets are of interest:

$$\text{FEA}\,(q, M) = \{x : x \ \text{ satisfies (1) and (2)}\} \quad \text{ and}$$

$$\text{SOL}\,(q, M) = \{x : x \text{ satisfies (1), (2) and (3)}\}.$$

For $M \in \mathcal{R}^{n \times n}$ and every $\alpha \subseteq \{1, \ldots, n\}$ there is a corresponding principal submatrix of M denoted $M_{\alpha\alpha}$. The determinant of a principal submatrix is called a *principal minor* of M. When the principal submatrix is nonsingular (principal minor is nonzero), there is a corresponding *principal pivotal transformation* $\wp_\alpha$ given by

$$\begin{bmatrix} M_{\alpha\alpha} & M_{\alpha\bar{\alpha}} \\ M_{\bar{\alpha}\alpha} & M_{\bar{\alpha}\bar{\alpha}} \end{bmatrix} \xrightarrow{\wp_\alpha} \begin{bmatrix} M_{\alpha\alpha}^{-1} & -M_{\alpha\alpha}^{-1} M_{\alpha\bar{\alpha}} \\ M_{\bar{\alpha}\alpha} M_{\alpha\alpha}^{-1} & M_{\bar{\alpha}\bar{\alpha}} - M_{\bar{\alpha}\alpha} M_{\alpha\alpha}^{-1} M_{\alpha\bar{\alpha}} \end{bmatrix}.$$

For $x \in \mathcal{R}^n$, the index set

$$\text{supp}\,(x) = \{i : x_i \neq 0\}$$

is called the *support* of x. Clearly $\text{supp}\,(x) = \emptyset$ if and only if $x = 0$. The definition of one of the matrix classes, $\mathbf{P}_*(\kappa)$, given below requires the index sets

$$I_+(x) = \{i : x_i > 0\} \quad \text{ and } \quad I_-(x) = \{i : x_i < 0\}$$

both of which are subsets of $\text{supp}\,(x)$.

3 List of matrix classes

See Table 1.

Table 1 List of matrix classes

$M \in$	if and only if	Common name	Source	Note
A	$M \in \mathbf{RA} \cap \mathbf{CA}$.	Adequate	[21]	
BG	$M_{\alpha\alpha} = 0,\ M_{\bar{\alpha}\bar{\alpha}} = 0,\ M_{\alpha\bar{\alpha}} > 0,\ M_{\bar{\alpha}\alpha} > 0$.	Bimatrix game	[28]	N[a]
CA	$M \in \mathbf{P_0}$ and $\forall\alpha,\ M_{\bullet\alpha}$ has linearly independent columns only if $\det M_{\alpha\alpha} > 0$.	Column adequate	[21]	N[b]
CSU	$x_i(Mx)_i \leq 0 \quad \forall i \Longrightarrow x_i(Mx)_i = 0 \quad \forall i$.	Column sufficient	[11]	
C	$x^{\mathrm{T}}Mx > 0,\ \forall$ nonzero $x \geq 0$.	Strictly copositive	[29]	
$\mathbf{C_0}$	$x^{\mathrm{T}}Mx \geq 0,\ \forall x \geq 0$.	Copositive	[29]	
$\mathbf{C_0^+}$	$M \in \mathbf{C_0}$ and $[\,x^{\mathrm{T}}Mx = 0,\ x \geq 0\,] \Longrightarrow (M + M^{\mathrm{T}})x = 0$.	Copositive-plus	[26]	N[c]
$\mathbf{C_0^*}$	$M \in \mathbf{C_0}$ and $x \in \mathrm{SOL}\,(0, M) \Longrightarrow M^{\mathrm{T}}x \leq 0$.	Copositive-star	[19]	
D	$M \in \bigcup_{d>0} \mathbf{L(d)}$.		[18]	N[d]
$\mathbf{D^*}$	$M \in \bigcup_{d>0} \mathbf{L^*(d)}$.		[18]	
E	$\forall x \geq 0,\ x \neq 0,\ \exists i : x_i > 0,\ (Mx)_i > 0$.	Strictly semimonotone	[9]	N[e]
$\mathbf{E^*}$	$\mathrm{SOL}\,(q, M) = \{0\}$ for all nonzero $q \geq 0$.	Strongly semimonotone	[13]	
$\mathbf{E'}$	$M \in \mathbf{E^*} \setminus \mathbf{E}$.		[14]	
E(d)	if $z \in \mathrm{SOL}\,(d, M),\ \exists$ nonzero $x \geq 0$ such that $y := -M^{\mathrm{T}}x \geq 0$ and $z \geq x,\ w := d + Mz \geq y$.		[18]	N[f]
$\mathbf{E^*(d)}$	$\mathrm{SOL}\,(d, M) = \{0\}$.		[18]	
$\mathbf{E_0}$	$\forall x \geq 0,\ x \neq 0,\ \exists i : x_i > 0,\ (Mx)_i \geq 0$.	Semimonotone		
$\mathbf{E_1}$	$\forall$ nonzero $x \in \mathrm{SOL}\,(0, M)\ \exists$ nonnegative diagonal matrices $\Gamma,\ \Omega$ such that $(\Gamma M + M^{\mathrm{T}}\Omega)x = 0$ and $\Omega x \neq 0$.		[15]	
G	$M \in \bigcup_{d>0} \mathbf{E(d)}$		[20]	N[g]
$\mathbf{G^\sharp}$	$M \in \mathbf{G}$ and $x \in \mathrm{SOL}\,(0, M) \Longrightarrow (M + M^{\mathrm{T}})x = 0$.		[20]	
GI	every column of M has at most one nonzero entry.		[33]	N[h]
GNI	$M \in \mathbf{GI}$ and $M \leq 0$.		[33]	
H	$\exists d > 0$ such that $\forall i$, $\mathrm{mod}\ m_{ii}d_i > \sum_{j\neq i} \mathrm{mod}\ m_{ij}d_j$.		[30]	N[i]
$\mathbf{H^+}$	$M \in H$ and $m_{ii} > 0\ \forall i$.			N[j]
INS	$\exists$ positive integer k such that $\forall q \in \mathrm{int}\,K(M)$, (q, M) has exactly k solutions.		[33]	
K	$M \in \mathbf{P} \cap \mathbf{Z}$.		[17]	N[k]

Table 1 continued

$M \in$	if and only if	Common name	Source	Note
$\mathbf{K_0}$	$M \in \mathbf{P_0} \cap \mathbf{Z}$.		[17]	
$\mathbf{K^h}$	$M \in \mathbf{P} \cap \mathbf{Z^h}$.	Hidden-**K**		
$\mathbf{L}$	$M \in \mathbf{E_0} \cap \mathbf{E_1}$.		[15]	$\mathrm{N^l}$
$\mathbf{L_*}$	$M \in \mathbf{E}$.	Strictly semimonotone	[13]	
$\mathbf{L^*}$	SOL $(q, M)=\{0\}$, for all nonzero $q \geq 0$.		[13]	
$\bar{\mathbf{L}}$	$M_{\alpha\alpha} \in \mathbf{L},\ \forall\alpha$.	Completely-**L**	[2]	
$\mathbf{L^f}$	All principal pivotal transforms of M belongs to **L**.	Fully-**L**	[2]	
$\mathbf{L(d)}$	$\mathbf{E(d)} \cap \mathbf{E(0)}$.		[18]	
$\mathbf{L^*(d)}$	$\mathbf{E^*(d)} \cap \mathbf{E^*(0)}$		[18]	
$\mathbf{N}$	$\det M_{\alpha\alpha} < 0,\ \forall\alpha$.		[32]	$\mathrm{N^m}$
$\mathbf{P}$	$\det M_{\alpha\alpha} > 0,\ \forall\alpha$.			$\mathrm{N^n}$
$\mathbf{P_0}$	$\det M_{\alpha\alpha} \geq 0,\ \forall\alpha$.			
$\mathbf{P_1}$	$M \in \mathbf{P_0}$ and $\det M_{\alpha\alpha} = 0$ for just one α.		[33]	$\mathrm{N^o}$
$\mathbf{P_*}(\kappa)$	$(1+4\kappa)\sum_{i\in I_+(x)} x_i(Mx)_i + \sum_{i\in I_-(x)} x_i(Mx)_i \geq 0,\ \forall x$.		[25]	
$\mathbf{P_*}$	$\bigcup_{\kappa\geq 0} \mathbf{P_*}(\kappa)$.		[25]	$\mathrm{N^p}$
PD	$x^{\mathrm{T}}Mx > 0,\ \forall x \neq 0$.	Positive definite		
PSD	$x^{\mathrm{T}}Mx \geq 0,\ \forall x$.	Positive semidefinite		$\mathrm{N^q}$
$\mathbf{Q}$	SOL $(q, M) \neq \emptyset,\ \forall q$.			$\mathrm{N^r}$
$\mathbf{Q_0}$	$\forall q$ FEA $(q, M) \neq \emptyset \Longrightarrow$ SOL $(q, M) \neq \emptyset$.			$\mathrm{N^s}$
$\bar{\mathbf{Q}}$	$M_{\alpha\alpha} \in \mathbf{Q},\ \forall\alpha$.	Completely-**Q**	[7]	$\mathrm{N^t}$
$\mathbf{R}$	$\exists d > 0$ such that SOL $(\tau d, M) = \{0\}\ \forall\tau \geq 0$.	Regular	[23]	
$\mathbf{R_0}$	SOL $(0, M) = \{0\}$.		[18]	
$\mathbf{R_1}$	SOL (q, M) is bounded $\forall q \in$ int pos $[-M\ I]$.		[1]	
$\mathbf{R_2}$	SOL (q, M) is bounded $\forall q \in$ int $K(M)$.		[1]	
RA	$M^{\mathrm{T}} \in \mathbf{CA}$.	Row adequate	[21]	$\mathrm{N^u}$
RSU	$M^{\mathrm{T}} \in \mathbf{CSU}$.	Row sufficient	[11]	
$\mathbf{S}$	$\exists x \geq 0$ such that $Mx > 0$.			
$\mathbf{S_0}$	$\exists x \geq 0,\ x \neq 0$ such that $Mx \geq 0$.			
SU	$M \in \mathbf{RSU} \cap \mathbf{CSU}$.	Sufficient	[11]	
$\mathbf{T}$	M has property (T).		[4]	$\mathrm{N^v}$
$\mathbf{T_*}$	FEA $(0, -M^{\mathrm{T}})$ = SOL $(0, -M^{\mathrm{T}})$.		[1]	
$\mathbf{U}$	$q \in$ int $K(M) \Longrightarrow (q, M)$ has a unique solution.		[33]	$\mathrm{N^w}$
$\mathbf{V}$	$\forall\alpha,\ \nexists x_\alpha > 0$ such that $M_{\alpha\alpha}x_\alpha = (0, \ldots, 0, \xi)^{\mathrm{T}},\ \xi \leq 0$.		[35]	$\mathrm{N^x}$
$\mathbf{W}$	$\forall\alpha$, pos $C_M(\alpha) \cap$ pos $C_M(\bar{\alpha}) = \{0\}$.		[22]	
$\mathbf{Y}$	a generic matrix class.			
$\bar{\mathbf{Y}}$	every principal submatrix of M belongs to **Y**.	Completely-**Y**		
$\mathbf{Y^f}$	every principal pivotal transform of M belongs to **Y**.	Fully-**Y**		
$\mathbf{Y^s}$	if $M \in \mathbf{Y}$ then $SMS^{\mathrm{T}} \in \mathbf{Y}$.	Sign-change invariant **Y**	[2]	
$\mathbf{Z}$	$m_{ij} \leq 0$ for all $i \neq j$.		[17]	

Table 1 continued

$M \in$	if and only if	Common Name	Source	Note
$\mathbf{Z^+}$	$M \in \mathbf{Z}$ and $m_{ii} \geq 0$ for all i.			N[y]
$\mathbf{Z^h}$	$\exists\, X \in \mathbf{Z},\ Y \in \mathbf{Z}$ and nonnegative vectors r and s such that $MX = Y$, and $r^{\mathrm{T}}X + s^{\mathrm{T}}Y > 0$.	Hidden-$\mathbf{Z}$		

[a] It is not restrictive to assume $M_{\alpha\bar{\alpha}} > 0,\ M_{\bar{\alpha}\alpha} > 0$

[b] The conditions for membership in this class are due to Ingleton [21]. The name for it is due to Eaves [15,16]

[c] Lemke used this class in [26], but he did not give it a name or a notation. The name copositive plus first appeared in [9, p. 116]. The notation $\mathbf{C_0^+}$ came later

[d] The notations **D** and **D*** are "new" ad hoc ways of representing classes that have a long history

[e] This class appears in [9] (with no name or notation) and in [15] (where it is denoted $\mathcal{L}_*$). The name strictly semimonotone was introduced by Karamardian [23]

[f] The vector d is arbitrary although the case where $d > 0$ is of special interest. Garcia [18] notes that $\mathbf{E(0)} = \mathcal{L}_2$ which is now denoted $\mathbf{E_1}$

[g] The symbol used for this class refers to C.B. Garcia who introduced it (without the notation) in [18]

[h] It appears that this matrix class does not lie within any interesting matrix class. See Stone [33, p. 142]

[i] See [12, **3.13.7**]

[j] Such matrices are diagonally stable and belong to **P**. See [12, **3.3.15**]. The notation $\mathbf{H^+}$ may originate in this document

[k] At one time, the notation **K** was used for what is now denoted by $\mathbf{Q_0}$. See for example, [31, p. 576]. The wealth of information in [17] did much to influence the change

[l] See also [16]. Eaves used the calligraphic notation $\mathcal{L}$ for what we denote as **L**. Moreover, he defined $\mathcal{L}$ as $\mathcal{L}_1 \cap \mathcal{L}_2$ whereas in our notation $\mathcal{L}_1 = \mathbf{E_0}$ and $\mathcal{L}_2 = \mathbf{E_1}$. It may have been that Eaves was honoring Lemke, and vice versa

[m] See also [24] and [33]

[n] The first use of this class in the LCP context is believed to be in [5], a portion of which is published in [6] (Although [5] and [6] have the same title, the article is extracted from the thesis.)

[o] The first appearance of this class in the periodical literature occurs in [10]

[p] This class equals **SU**. See [34]

[q] Positive semidefinite matrices are sometimes called *monotone*

[r] Garcia [18] was the first to use the notation **Q** for this class

[s] The class $\mathbf{Q_0}$ was defined by Parsons [31], but he denoted it **K**

[t] This class is known to be the same as **E** (strictly semimonotone matrices) and **V**

[u] The conditions for membership in this class are due to Ingleton [21]. The name for it is due to Eaves [15,16]

[v] An $n \times n$ matrix M has property (T) if for all $\alpha \subseteq \{1, \ldots, n\}$ the existence of a solution to the linear inequality system $M_{\alpha\alpha}z_\alpha \leq 0,\quad M_{\bar{\alpha}\alpha}z_\alpha \geq 0,\ z_\alpha > 0$ implies the existence of a solution to the system $y_{\alpha_0}^{\mathrm{T}}M_{\alpha\alpha} = 0,\ y_{\alpha_0}^{\mathrm{T}}M_{\alpha_0\bar{\alpha}} \leq 0, 0 \neq y_{\alpha_0} \geq 0$ where $\alpha_0 = \{i \in \alpha : M_{i\alpha}z_\alpha = 0\}$

[w] See also [10]

[x] It was shown in [7] that $\mathbf{V} = \mathbf{E} = \bar{\mathbf{Q}}$

[y] Stone [33] used the notation $\bar{\mathbf{Z}}$ for this class, but nowadays that notation would signify completely-**Z** (although it must be conceded that **Z** is a complete class so that $\bar{\mathbf{Z}} = \mathbf{Z}$). Stone noted that what his $\bar{\mathbf{Z}}$ had been denoted **L**

4 Structural properties

See Table 2.

Table 2 Complete, Full, Reflective, and Sign-invariant matrix classes in the linear complementarity problem

Class	Complete	Full	Reflective	Sign-invariant
A	Yes[a]	No	Yes	Yes
BG	Yes	No	Yes	No
C	Yes	No	Yes	No
CA	Yes	No	No	Yes
CSU	Yes	Yes[b]	No	Yes
$\mathbf{C_0}$	Yes	No	Yes	No
$\mathbf{C_0^+}$	Yes	No	Yes	No
E	Yes	No	Yes	No
$\mathbf{E_0}$	Yes	No	Yes	No
$\mathbf{E_1}$	No	Yes	No	No[c]
K	Yes	No	Yes	No
$\mathbf{K_0}$	Yes	No	Yes	No
N	Yes	No	Yes	Yes
P	Yes	Yes	Yes	Yes
$\mathbf{P_0}$	Yes	Yes	Yes	Yes
$\mathbf{P_1}$	No	Yes	Yes	Yes
$\mathbf{P_*}$	Yes	Yes	Yes	Yes
PD	Yes	Yes	Yes	Yes
PSD	Yes	Yes	Yes	Yes
Q	No	Yes	No	No
$\mathbf{Q_0}$	No	Yes	No	No
RA	Yes	No	No	Yes
RSU	Yes	Yes[d]	No	Yes
Z	Yes	No	Yes	No
$\mathbf{Z^+}$	Yes	No	Yes	No

Complete: Every principal submatrix of every member of the class is a member of the class. *Full*: Every principal pivotal transform of every member of the class is a member of the class. *Reflective*: The transpose of every member of the class is a member of the class; *Sign-invariant*: For every member M of the class, and every conformable sign-changing matrix S, the matrix SMS belongs to the class. The property $S^2 = I$ implies $S = S^{-1}$, whence $M \to SMS$ is a *similarity transformation*

[a] Ingleton [21, p. 522] notes this property in (1.2.4) but does not use our term for it. A few lines below (1.2.4) he introduces Definition 1.3 to the effect that a matrix is *completely adequate* if it is adequate with respect to every orthonormal basis

[b] See [8]

[c] The observation that this class is not sign-change invariant (see the list in Sect. 4) is due to Adler and Wong [3]

[d] See [8]

5 Matrix class inclusion map

See Fig. 1.

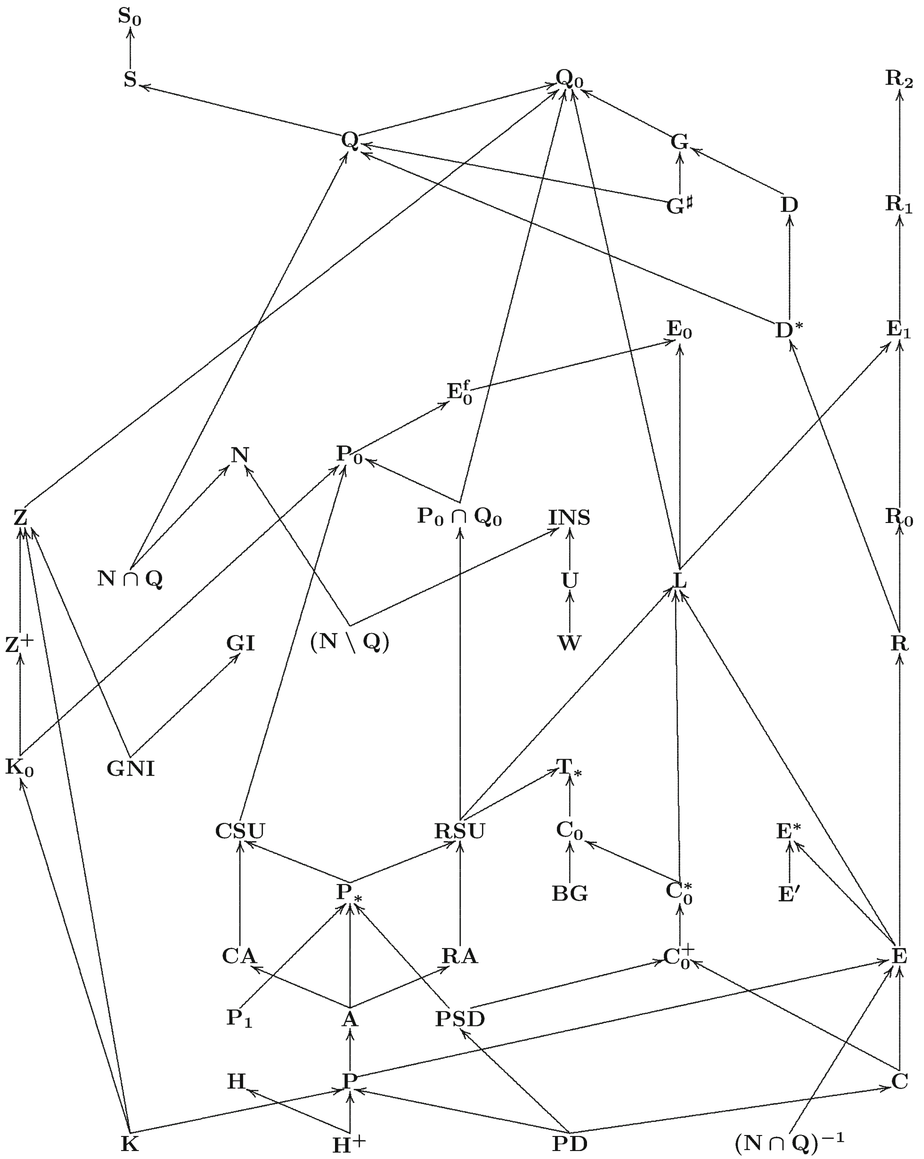

Fig. 1 Matrix class inclusion map. Arrows for implied inclusions are omitted

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